The harmonic analysis of kernel functions

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Joint work with:

A. Chiuso (University of Padova)

Kernels in system identification

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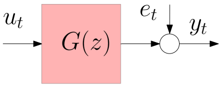
Model class (e.g. OE models)

$$y_t = \sum_{s=1}^{\infty} g_s u_{t-s} + e_t \quad g_t \text{ impulse response}$$



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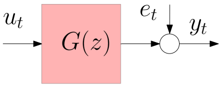

Gaussian linear regression model

$$y^N = \Phi\theta + e^N$$

$$\theta \sim \mathcal{N}(0, K), \quad K \text{ kernel function}$$

$$y^N := \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \; \theta := \begin{bmatrix} g_1 \\ g_2 \\ \vdots \end{bmatrix}$$

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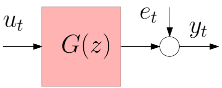

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g_t is modeled as Gaussian process with zero mean and covariance function $\mathrm{Cov}[g_t g_s] = K(t, s)$

Kernels in system identification (cont'd)

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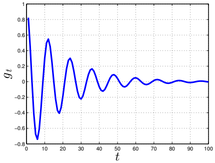
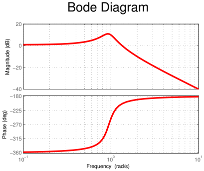
Our a priori information on the impulse response:

- BIBO stable
- Frequency content





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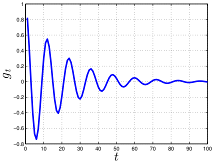

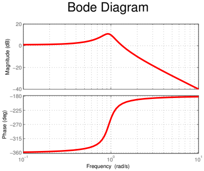

Question

How to embed this information in the kernel function?

Modeling the impulse response

How?

Modeling the impulse response

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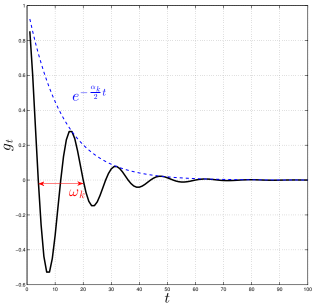
g_t as a sum of damped sinusoids

$$g_t = \sum_{k=1}^{M} |c_k| e^{-\frac{\alpha_k}{2}t} \cos(\omega_k t + \angle c_k)$$



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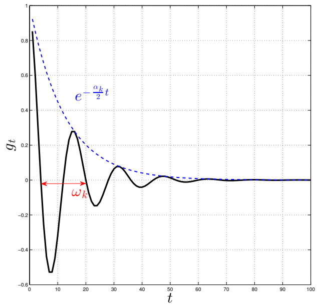

- c_k complex Gaussian random variable such that:
  - c_k is zero mean
  - $\mathrm{Cov}(c_k, \bar{c}_j) = p_k \delta_{k-j}$
  - $\mathrm{Cov}(c_k, c_j) = 0$

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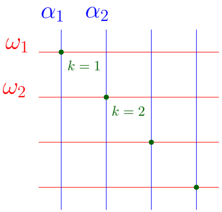

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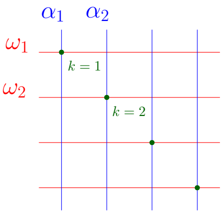

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"Infinite dense sum" of damped sinusoids

$$g_t = \int_0^{\infty} \int_{-\infty}^{\infty} |c(\alpha, \omega)| e^{-\frac{\alpha}{2} t} \cos(\omega t + \angle c(\alpha, \omega)) \mathrm{d}\omega \mathrm{d}\alpha$$

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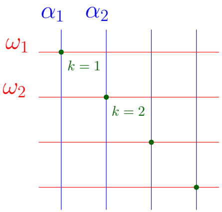

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$c(\alpha,\omega)$ generalized Fourier transform of g_t

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Harmonic representation of the kernel function K

$$K(t,s) = \frac{1}{2}\int_0^\infty \int_{-\infty}^\infty p(\alpha,\omega) e^{-\alpha\frac{t+s}{2}} \cos(\omega(t-s)) \mathrm{d}\omega \mathrm{d}\alpha$$

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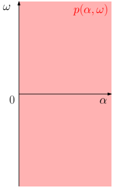

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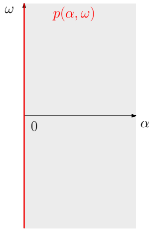
Choice: $p(\alpha, \omega) = \delta(\alpha)q(\omega)$



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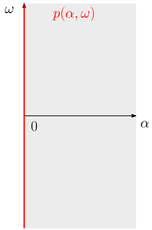

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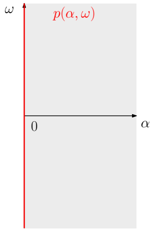

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Stationary process ("infinite dense sum" of sinusoids)

$$g_t = \int_{-\infty}^{\infty} |c(\omega)|\cos(\omega t + \angle c(\omega))\mathrm{d}\omega$$ $c(\omega)$ Fourier transform

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Shape	$\tilde{q}(\omega)$	Kernel
Laplacian	$\tilde{q}_L(\omega) = \frac{\beta/2}{\pi[(\omega-\omega_0)^2+(\beta/2)^2]}$	$K_L(t-s) = e^{-\frac{\beta}{2}\lvert t-s\rvert}\cos(\omega_0(t-s))$
Cauchy	$\tilde{q}_C(\omega) = \frac{1}{2\beta}e^{-\frac{\lvert\omega-\omega_0\rvert}{\beta}}$	$K_C(t-s) = \frac{1}{1+\beta^2(t-s)^2}\cos(\omega_0(t-s))$
Gaussian	$\tilde{q}_G(\omega) = \frac{1}{\sqrt{2\pi\beta^2}}e^{-\frac{(\omega-\omega_0)^2}{2\beta^2}}$	$K_G(t-s) = e^{-\frac{\beta^2(t-s)^2}{2}}\cos(\omega_0(t-s))$

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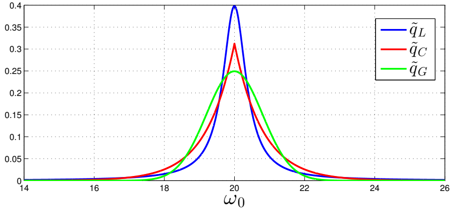

ω_0 center frequency

β bandwidth

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Locally stationary kernel (Silverman, 1957)

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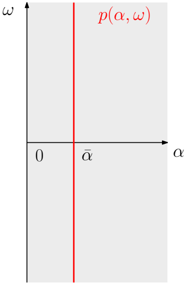

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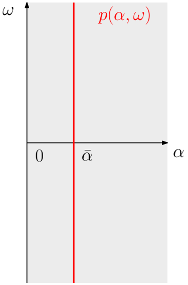

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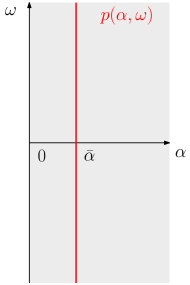

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- Examples of ECLS kernels (Chen-Ljung, 2015): stable-spline, dynamic-correlated, tuned-correlated...

Example 2: Locally stationary kernels (cont'd)

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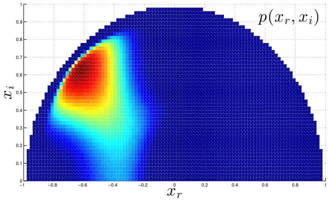
Probability density function of a 2^{nd} order stable model

- Transfer function

$$G(z) = \frac{1}{(z-\rho)(z-\bar{\rho})} = \sum_{k=1}^{\infty} g_k z^{-k} \quad |\rho| < 1$$

- $\rho = x_r + jx_i$ pole of the model
- g_k process with kernel $K \rightarrow g_k$ with pdf $p(x_r, x_i)$



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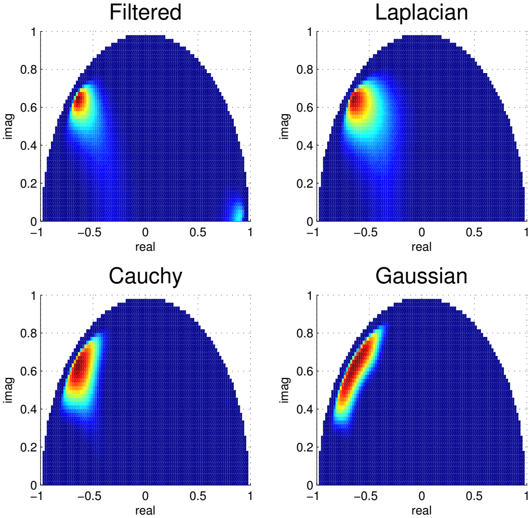

- $\bar{\alpha} = 0.9$
- $\omega_0 = \frac{3}{4}\pi$
- β small

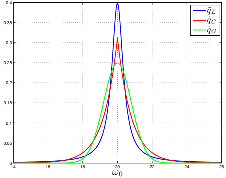

Figure: pdf of a 2^{nd} order stable model with ECLS

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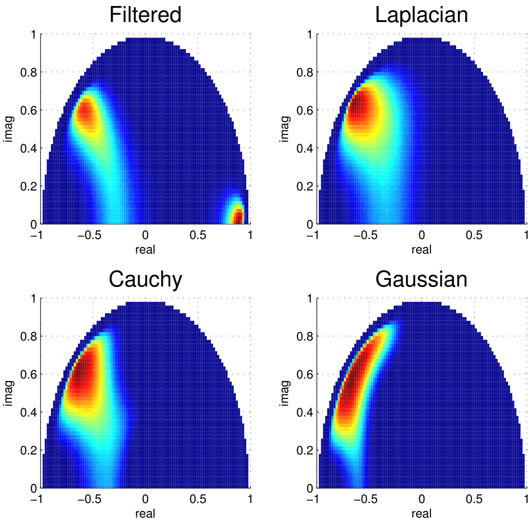

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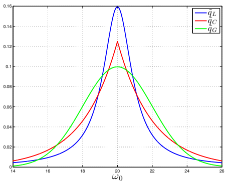

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$$p(\alpha,\omega) = 1_{[\alpha_m,\alpha_M]}(\alpha)\frac{\alpha/2}{\pi[\omega^2+(\alpha/2)^2]}$$

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Integrated TC kernel (Pillonetto *et. al.*)

$$K(t, s) = \frac{e^{-\alpha_m \max\{t,s\}} - e^{-\alpha_M \max\{t,s\}}}{\max\{t, s\}}$$

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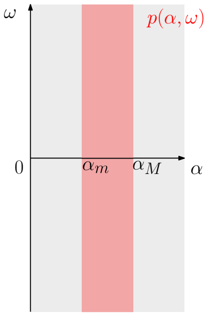

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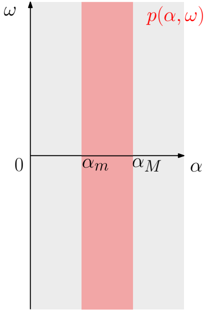

Integrated kernel

$$K(t,s) = 2\frac{e^{-\alpha_m \frac{t+s}{2}} - e^{-\alpha_M \frac{t+s}{2}}}{t+s} \underbrace{K(t-s)}_{\text{stationary kernel}}$$

Example 3: Integrated kernels (cont'd)

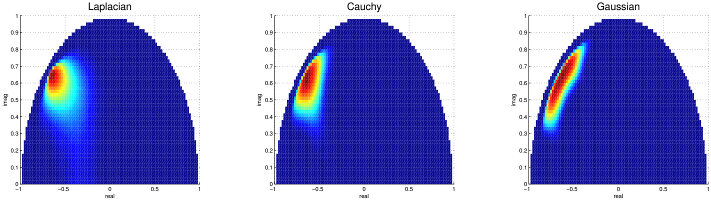

Figure: pdf of a 2^{nd} order model with ECLS kernel

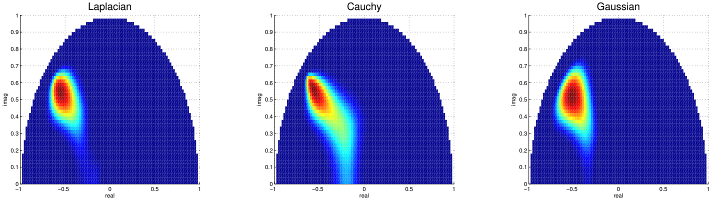

Figure: pdf of a 2^{nd} order model with INTEGRATED kernel

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